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ELECTROELASTIC MODELLING OF ANISOTROPIC PIEZOELECTRIC MATERIALS WITH AN ELLIPTIC INCLUSION

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Abstract—Theoretical work concerning the microstructural characterization and analysis of anisotropic piezoelectric media with inclusions is rather scarce, especially in comparison with numerous results and micromechanics models which exist today on the inclusion problem in anisotropic elastic media. In this investigation, the Stroh formalism is used to develop a general solution for an infinite, anisotropic piezoelectric medium with an elliptic inclusion; the coupled elastic and electric fields both inside the inclusion and on the boundary of the inclusion and matrix are given.

1. INTRODUCTION

For many years piezoelectric ceramics and composites have served well as functional elements for converting electrical energy into mechanical energy, and vice versa. Generally, a piezoelectric material is a complex system composed of crystallites, defects (cracks or pores) and inclusions (piezoelectric ceramic fibres or particles). The existence of these defects and inclusions greatly affects the electric, dielectric, piezoelectric, elastic and mechanical properties of such materials (Okazaki, 1985). To gain an understanding of the piezoelectric effect in ceramics we must first consider the behaviour of the material on a microscopic scale and determine the effects of defects and inclusions on the properties of such materials. The electroelastic analysis and effective constant predication of piezoelectric materials with defects and inclusions become a very important topic in the design and manufacturing of piezoelectric elements.

The coupled electroelastic behaviour of the constituents presents a level of difficulty not present in the design and analysis of the mechanical behaviour of the piezoelectric materials. Further complicating factors arise from the inherent anisotropy of the piezoelectric ceramics. Nevertheless, a reasonable amount of theoretical work has been directed towards the study of the dislocation, crack and inclusion problems and interfacial behaviour in homogeneous piezoelectric solids [see, for example, Deeg (1980), Zhou *et al.* (1986), Sosa and Pak (1990), Benveniste (1992), Suo *et al.* (1992), Wang (1992a, b), Chen (1993a, b), Dunn and Taya (1993)]. Sosa and Pak (1990) developed a three-dimensional solution for isotropic piezoelectric ceramic with defects. Wang (1992a, b) and Chen (1993a, b) examined the inclusion and crack problems in a piezoelectric matrix based on the Green function and Fourier transformation techniques. By reexamination of Deeg's (1980) rigorous analytical solution for a piezoelectric inclusion, Dunn and Taya (1993) estimated the effective properties using the dilute, self-consistent, Mori–Tanaka and differential micromechanical models. For a wide survey on micromechanical modelling of piezocomposites and the determination of their effective moduli, the reader is referred to recent papers by Benveniste (1993a, b), Maugin *et al* (1992), and Avellaneda and Olson (1993).

The present paper is concerned with deriving an exact general solution for an infinite, anisotropic piezoelectric medium with an elliptic inclusion; the coupled electroelastic fields of the inclusion and matrix are given when the external elastic field and electric field are constant. The developed theory is based on the central idea of the Stroh formalism established by Stroh (1962) and further elaborated by Ting (1986, 1988). In contrast to the classical formulation, the Stroh formalism has been proved to be elegant and powerful in solving two-dimensional anisotropic elasticity problems. More recently, the Stroh formalism has been generalized to treat dislocations and line charges in linear piezoelectric media by Pak (1992) and solve the boundary value problems of electroelastic media by Suo *et al.* (1992). In the following sections we consider two-dimensional elliptic piezoelectric inclusion problems and use the Stroh formalism to find the coupled elastic and electric fields inside the inclusion and the matrix interfacial quantities. Universal relations are derived between the coupled fields and the effective elastic, piezoelectric and dielectric constants of the piezoelectric solids. These relations are shown to reduce to Hwu and Ting's result in the special case of anisotropic nonpiezoelectric media (Hwu and Ting, 1989).

2. BASIC EQUATIONS

If the free charges and body forces do not exist in a piezoelectric body, the static elastic and electric field equations can be written as (Maugin, 1988):

$$\sigma_{ij,i} = 0 \tag{1}$$

$$D_{i,i} = 0 \tag{2}$$

$$\sigma_{ij} = C_{ijrs}\gamma_{rs} - e_{sji}E_s \tag{3}$$

$$D_i = \varepsilon_{is}E_s + e_{irs}\gamma_{rs}, \tag{4}$$

where γ, σ, $\mathbf{D}$ and $\mathbf{E}$ are the strain, stress, the electric displacement and the electric field, respectively. The elastic, piezoelectric and dielectric constants of the medium are represented by the fourth, third and second order tensors $\mathbf{C}$, $\mathbf{e}$ and ε respectively, which satisfy the symmetry relations:

$$C_{ijrs} = C_{jirs} = C_{ijsr} = C_{srji}, \quad e_{irs} = e_{isr}, \quad \varepsilon_{is} = \varepsilon_{si}. \tag{5}$$

If $\mathbf{u}$ is the elastic displacement vector and $\mathbf{\Phi}$ the electric potential, the infinitesimal strain γ and the electric field E are derived from gradients:

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad E_i = -\mathbf{\Phi}_{,i}. \tag{6}$$

Substitution of eqns (6) into eqns (3) and (4) yields

$$\sigma_{ij} = C_{ijrs}u_{r,s} + e_{sji}\Phi_{,s}, \quad D_i = -\varepsilon_{is}\Phi_{,s} + e_{irs}u_{r,s}. \tag{7}$$

Inserting eqns (7) into (1) and (2), respectively, we obtain

$$(C_{ijrs}u_r + e_{sji}\Phi)_{,si} = 0, \quad (-\varepsilon_{is}\Phi + e_{irs}u_r)_{,si} = 0. \tag{8}$$

For two-dimensional problems in which u and Φ depend on x_1 and x_2 only, the general solution can be obtained by considering an arbitrary function of a linear combination of x_1 and x_2:

$$\{u_r, \Phi\} = \mathbf{a}f(\xi_1x_1 + \xi_2x_2). \tag{9}$$

It is convenient, here and in the sequel, to take $\{u_r, \Phi\}$ to be a column with the entities indicated, so that $\mathbf{a}$ is likewise a four-component column. Without loss of generality, one can always take $\xi_1 = 1$, $\xi_2 = p$. Thus, the number p and the column $\mathbf{a}$ are determined by substituting eqn (8) into (9), which gives

$$(C_{\alpha jr\beta}a_r + e_{\alpha j\beta}a_4)\xi_\alpha\xi_\beta = 0\,, \quad (-\varepsilon_{\alpha\beta}a_4 + e_{\alpha r\beta}a_r)\xi_\alpha\xi_\beta = 0, \tag{10}$$

where $\alpha, \beta = 1$ or 2. This is an eigenvalue problem consisting of four equations; a nontrivial **a** exists if p is a root of the determinant polynomial. Since eqn (10) admits no real root (Suo *et al.*, 1992) and the $\mathbf{p}_\alpha$ occur as four pairs of complex conjugates, we let

$$p_{\alpha+4} = \bar{p}_\alpha\,, \quad \mathrm{Im}(p_\alpha) > 0\,, \quad \alpha = 1,2,3,4, \tag{11}$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part. More generally, we have

$$V = \{u_r\,, \quad \Phi\} = 2\mathrm{Re}\sum_{\alpha=1}^{4} \mathbf{a}_\alpha f_\alpha(z_\alpha), \tag{12}$$

in which Re stands for the real part, $\mathbf{a}_\alpha$ the associated columns, and $z_\alpha = x_1 + p_\alpha x_2$. For a given boundary value problem, the stress and the electric displacement obtained from eqns (7) and (12) are given by

$$\{\sigma_{2j}\,, \quad D_2\} = 2\mathrm{Re}\sum_{\alpha=1}^{4} \mathbf{b}_\alpha f'_\alpha(z_\alpha)\,, \quad \{\sigma_{1j}\,, \quad D_1\} = -2\mathrm{Re}\sum_{\alpha=1}^{4} \mathbf{b}_\alpha f'_\alpha(z_\alpha), \tag{13}$$

where, for a pair $(p, \mathbf{a})$, the associated **b** is

$$b_j = (C_{2jr\beta}a_r + e_{\beta j2}a_4)\xi_\beta\,, \quad b_4 = (-\varepsilon_{2\beta}a_4 + e_{2r\beta}a_r)\xi_\beta. \tag{14}$$

Substituting eqn (14) into (10) gives an alternative expression:

$$b_j = -p^{-1}(C_{1jr\beta}a_r + e_{\beta j1}a_4)\xi_\beta\,, \quad b_4 = -p^{-1}(-\varepsilon_{1\beta}a_4 + e_{1r\beta}a_r)\xi_\beta. \tag{15}$$

In matrix notation, eqns (14) and (15) are expressed as

$$\mathbf{b} = (R^{\mathrm{T}} + pT)\mathbf{a} = -\frac{1}{p}(Q + pR)\mathbf{a}, \tag{16}$$

where the superscript T is the transpose and Q, R and T are 4×4 matrices given by

$$\left.\begin{aligned} R &= \begin{bmatrix} C_{1jr2} & e_{2j1} \\ e_{1r2}^{\mathrm{T}} & -\varepsilon_{12} \end{bmatrix}_{4\times4} \\ Q &= \begin{bmatrix} C_{1jr1} & e_{1j1} \\ e_{1r1}^{\mathrm{T}} & -\varepsilon_{11} \end{bmatrix}_{4\times4} \\ T &= \begin{bmatrix} C_{2jr2} & e_{2j2} \\ e_{2r2}^{\mathrm{T}} & -\varepsilon_{22} \end{bmatrix}_{4\times4} \end{aligned}\right\}. \tag{17}$$

We see that Q and T are symmetric and T is positive definite, so eqn (16) can be recast in the standard eigenrelation

$$N\xi = p\xi \tag{18}$$

$$N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^{\mathrm{T}} \end{bmatrix}, \quad \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{19}$$

$$\left.\begin{aligned} N_1 &= -T^{-1}R^{\mathrm{T}}, N_2 = T^{-1} = N_2^{\mathrm{T}} \\ N_3 &= RT^{-1}R^{\mathrm{T}} - Q = N_3^{\mathrm{T}} \end{aligned}\right\}, \tag{20}$$

where N_2 and N_3 are also symmetric and N_2 is positive definite. If we define the 4×4 matrices A and B by

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4), \quad B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4), \tag{21}$$

each $\mathbf{a}$ is determined by the eigenvalue problem up to a complex-valued normalization constant. Assuming that the eigenvalues p_α are distinct or, if there is a repeated eigenvalue, the associated eigenvectors $\mathbf{a}_\alpha$ are independent of each other, it can be proved that the matrices H, L, S introduced by Barnett and Lothe (1973):

$$\left.\begin{aligned} S &= i(2AB^{\mathrm{T}} - I) \\ H &= 2iAA^{\mathrm{T}} \\ L &= -2iBB^{\mathrm{T}} \end{aligned}\right\}, \tag{22}$$

where $i = \sqrt{-1}$ and I is the unit matrix, are real and valid for the coupled electroelastic problem in the Appendix. Moreover, H and L are symmetric and positive definite, the matrices $H^{-1}S$, SH, LS, SL^{-1} are antisymmetric, so we also have the identity

$$HL - SS = LH - S^{\mathrm{T}}S^{\mathrm{T}} = I, \quad LS + S^{\mathrm{T}}L = SH + HS^{\mathrm{T}} = 0. \tag{23}$$

By introducing the auxiliary function as follows:

$$U = 2\mathrm{Re} \sum_{\alpha=1}^{4} \mathbf{b}_\alpha f_\alpha(z_\alpha), \tag{24}$$

eqns (13) can be rewritten as

$$\{\sigma_{2j}, D_2\} = U_{,1}, \quad \{\sigma_{1j}, D_1\} = -U_{,2}. \tag{25}$$

Finally, from eqns (12), (18) and (24), we have the following differential equation:

$$\{V_{,2}, U_{,2}\} = N\{V_{,1}, U_{,1}\}. \tag{26}$$

3. ELECTRIC AND ELASTIC FIELDS IN MATRIX AND INCLUSION

In an infinite anisotropic piezoelectric material, consider an elliptic inclusion whose boundary is given by

$$x_1 = a\cos\psi \quad x_2 = b\sin\psi, \tag{27}$$

where $2a$ and $2b$ are the major and minor axes of the ellipse. The inclusion is trended indefinitely in the x_3-direction, the uniform stress and electric field are applied at infinity. The inclusion and the matrix have perfect bonding along the interface (27). Let $\sigma_{ij}^\infty, \gamma_{ij}^\infty$ be the stresses and strains, D_i^∞, E_s^∞ the electric displacement and electric field in the matrix at infinity. They are defined from eqns (3) and (4):

$$\sigma_{ij}^\infty = C_{ijrs}\gamma_{rs}^\infty - e_{sji}E_s^\infty, \quad D_i^\infty = \varepsilon_{is}E_s^\infty + e_{irs}\gamma_{rs}^\infty. \tag{28}$$

We note that σ_{ij}^∞ have to be prescribed in such a way that $\gamma_{33}^\infty = 0$, so the auxiliary functions V^∞ and U^∞ can be expressed using the variables in an infinite body

$$V^{\alpha} = \{u_r^{\alpha}, \mathbf{\Phi}^{\alpha}\} = \{(x_1\gamma_1^{\alpha} + x_2\gamma_2^{\alpha}), (x_1E_1^{\alpha} + x_2E_2^{\alpha})\} \tag{29}$$

$$U^{\alpha} = \{(x_1t_2^{\alpha} - x_2t_1^{\alpha}), (x_1D_2^{\alpha} - x_2D_1^{\alpha})\}, \tag{30}$$

in which

$$\left.\begin{aligned} \gamma_1^{\alpha} &= \{\gamma_{11}^{\alpha}, 0, 2\gamma_{13}^{\alpha}\} = u_{,1}^{\alpha} \\ \gamma_2^{\alpha} &= \{2\gamma_{21}^{\alpha}, \gamma_{22}^{\alpha}, 2\gamma_{23}^{\alpha}\} = u_{,2}^{\alpha} \\ t_1^{\alpha} &= \{\sigma_{11}^{\alpha}, \sigma_{12}^{\alpha}, \sigma_{13}^{\alpha}\} \\ t_2^{\alpha} &= \{\sigma_{21}^{\alpha}, \sigma_{22}^{\alpha}, \sigma_{23}^{\alpha}\} \end{aligned}\right\}. \tag{31}$$

In many applications, including the present one, f_1, f_2, f_3, f_4 have the same functional form

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad \alpha \text{ not summed}, \tag{32}$$

where q_α, $\alpha = 1,2,3$ and 4, are arbitrary complex constants. If we introduce the diagonal matrices

$$z = \operatorname{diag}\{z_1, z_2, z_3, z_4\} \tag{33}$$

$$F(z) = \operatorname{diag}\{f(z_1), f(z_2), f(z_3), f(z_4)\}, \tag{34}$$

eqns (12) and (24) can be written as

$$V = 2\operatorname{Re}\{AF(z)\mathbf{q}\} \tag{35}$$

$$U = 2\operatorname{Re}\{BF(z)\mathbf{q}\}, \tag{36}$$

in which $\mathbf{q}$ is the 4×1 matrix whose elements are q_α, $\alpha = 1, 2, 3$ and 4. Before we superimpose the general solution (35) and (36) onto (29) and (30), we replace the complex constant $\mathbf{q}$ by

$$\mathbf{q} = A^{\mathrm{T}}\mathbf{g} + B^{\mathrm{T}}\mathbf{h}, \tag{37}$$

where $\mathbf{g}$ and $\mathbf{h}$ are real. We, therefore, consider the general solution

$$V = \{u_r, \mathbf{\Phi}\} = x_1\{\gamma_1^{\alpha}, E_1^{\alpha}\} + x_2\{\gamma_2^{\alpha}, E_2^{\alpha}\} + 2\operatorname{Re}\{AF(z)A^{\mathrm{T}}\}\mathbf{g} + 2\operatorname{Re}\{AF(z)B^{\mathrm{T}}\}\mathbf{h} \tag{38}$$

$$U = x_1\{t_2^{\alpha}, D_2^{\alpha}\} - x_2\{t_1^{\alpha}, D_1^{\alpha}\} + 2\operatorname{Re}\{BF(z)A^{\mathrm{T}}\}\mathbf{g} + 2\operatorname{Re}\{BF(z)B^{\mathrm{T}}\}\mathbf{h} \tag{39}$$

of the auxiliary function for the piezoelectric materials with elliptic inclusion. Following Lekhnitskii (1968), we choose the arbitrary function $f(z_\alpha)$ in $F(z)$ of the following form:

$$\left.\begin{aligned} F(z) &= \operatorname{diag}(\xi_1^{-1}, \xi_2^{-1}, \xi_3^{-1}, \xi_4^{-1}) \\ \xi_\alpha &= \{z_\alpha + [z_\alpha^2 - (a^2 + p_\alpha^2 b^2)]^{1/2}\}/(a - \mathrm{i}p_\alpha b) \end{aligned}\right\}. \tag{40}$$

It is clear that

$$\xi_\alpha^{-1} = \{z_\alpha - [z_\alpha^2 - (a^2 + p_\alpha^2 b^2)]^{1/2}\}/(a + \mathrm{i}p_\alpha b). \tag{41}$$

Along the interface (27), we then obtain

$$\xi_\alpha^{-1} = \cos\psi - \mathrm{i}\sin\psi, \quad F(z) = (\cos\psi - \mathrm{i}\sin\psi)I. \tag{42}$$

We next consider the general solution of auxiliary functions V^0 and U^0 in the inclusion of piezoelectric ceramics. According to Wang (1992a), the coupled elastic and electric fields inside the inclusion stay uniform when the external elastic field and electric field are constant for the piezoelectric medium containing an ellipsoidal inclusion. As a simple example, the above conclusion can be obtained for an elliptic inclusion

$$V^0 = \{u_r^0, \mathbf{\Phi}^0\} = \{(x_1\gamma_1^0 + x_2\gamma_2^0), (x_1E_1^0 + x_2E_2^0)\} \tag{43}$$

$$U^0 = \{(x_1t_2^0 - x_2t_1^0), (x_1D_2^0 - x_2D_1^0)\}, \tag{44}$$

where

$$\left.\begin{aligned} \gamma_1^0 &= \{\gamma_{11}^0, \omega, 2\gamma_{13}^0\} = u_{,1}^0 \\ \gamma_2^0 &= \{2\gamma_{21}^0 - \omega, \gamma_{22}^0, 2\gamma_{23}^0\} = u_{,2}^0 \\ t_1^0 &= \{\sigma_{11}^0, \sigma_{12}^0, \sigma_{13}^0\} \\ t_2^0 &= \{\sigma_{21}^0, \sigma_{22}^0, \sigma_{23}^0\} \end{aligned}\right\}, \tag{45}$$

in which ω is the rotation (counter clockwise) of the elliptic inclusion. The elastic and electric fields in the inclusion are also related by eqns (3) and (4)

$$\sigma_{ij}^0 = C_{ijrs}^0\gamma_{rs}^0 - e_{sji}^0E_s^0 \tag{46}$$

$$D_i^0 = \varepsilon_{is}^0E_s^0 + e_{irs}^0\gamma_{rs}^0, \tag{47}$$

where $C_{ijrs}^0, \varepsilon_{is}^0, e_{irs}^0$ are the elastic constants, dielectric permittivity and piezoelectric constants of the inclusion, respectively. From the basic solution given by (38) and (39) for the matrix and by (43), (44) for the inclusion, the problem reduces to the determination of the unknown constants $\mathbf{g}$, $\mathbf{h}$, (t_1^0, D_1^0),(t_2^0, D_2^0),(γ_1^0, E_1^0) and (γ_2^0, E_2^0) only. This is presented below.

Along the elliptic interface defined by (27), eqns (43) and (44) reduce to

$$V^0 = a\cos\psi\{\gamma_1^0, E_1^0\} + b\sin\psi\{\gamma_2^0, E_2^0\} \tag{48}$$

$$U^0 = a\cos\psi\{t_2^0, D_2^0\} - b\sin\psi\{t_1^0, D_1^0\}, \tag{49}$$

while (38) and (39) become, using (22) and (42),

$$V = \cos\psi(a\{\gamma_1^\infty, E_1^\infty\} + \mathbf{h}) + \sin\psi(b\{\gamma_2^\infty, E_2^\infty\} - S\mathbf{h} - H\mathbf{g}) \tag{50}$$

$$U = \cos\psi(a\{t_2^\infty, D_2^\infty\} + \mathbf{g}) - \sin\psi(b\{t_1^\infty, D_1^\infty\} - L\mathbf{h} + S^{\mathrm{T}}\mathbf{g}). \tag{51}$$

Assume the bond is perfect, so that the displacement and potential, the stress and electric displacement are continuous across the bonded segment:

$$V = V^0, \quad U = U^0. \tag{52}$$

This leads to

$$\{\gamma_1^0, E_1^0, t_2^0, D_2^0\} = \{\gamma_1^\infty, E_1^\infty, t_2^\infty, D_2^\infty\} + \frac{1}{a}\{\mathbf{h}, \mathbf{g}\} \tag{53}$$

$$\{\gamma_2^0, E_2^0, -t_1^0, -D_1^0\} = \{\gamma_2^\infty, E_2^\infty, -t_1^\infty, -D_1^\infty\} - \frac{1}{b}\begin{bmatrix} S & H \\ -L & S^{\mathrm{T}} \end{bmatrix}\{\mathbf{h}, \mathbf{g}\}. \tag{54}$$

From eqns (26), (31) and (45), we have

$$\{\gamma_2^\infty, E_2^\infty, -t_1^\infty, -D_1^\infty\} = N\{\gamma_1^\infty, E_1^\infty, t_2^\infty, D_2^\infty\} \tag{55}$$

$$\{\gamma_2^0, E_2^0, -t_1^0, -D_1^0\} = N^0\{\gamma_1^0, E_1^0, t_2^0, D_2^0\}, \tag{56}$$

in which N^0 is similar to that of eqns (19) and (20), but its material constants are that in the inclusion. According to the solution of anisotropic elasticity problems introduced by Hwu and Ting (1989), eqn (54) can be rewritten, using (53), (55) and (56), as

$$\begin{bmatrix} D_1 & D_2 \\ -D_3 & D_1^{\mathrm{T}} \end{bmatrix}\{\mathbf{h}, \mathbf{g}\} = b\{\mathbf{d}_1, \mathbf{d}_2\}, \tag{57}$$

where

$$\begin{aligned}
D_1 &= S + \frac{b}{a}N_1^0 \\
D_2 &= H + \frac{b}{a}N_2^0 \\
D_3 &= L - \frac{b}{a}N_3^0 \\
\mathbf{d}_1 &= (N_1 - N_1^0)\{\gamma_1^\infty, E_1^\infty\} + (N_2 - N_2^0)\{t_2^\infty, D_2^\infty\} \\
\mathbf{d}_2 &= (N_3 - N_3^0)\{\gamma_1^\infty, E_1^\infty\} + (N_1 - N_1^0)\{t_2^\infty, D_2^\infty\}.
\end{aligned}$$

Equation (57) can be solved for $\mathbf{h}$ and $\mathbf{g}$ explicitly by inverse transformation; we then obtain

$$\left.\begin{aligned}
\mathbf{h} &= b(D_3 + D_1^{\mathrm{T}} D_2^{-1} D_1)^{-1}(D_1^{\mathrm{T}} D_2^{-1}\mathbf{d}_1 - \mathbf{d}_2) \\
\mathbf{g} &= b(D_2 + D_1 D_3^{-1} D_1^{\mathrm{T}})^{-1}(\mathbf{d}_1 + D_1 D_3^{-1}\mathbf{d}_2)
\end{aligned}\right\}. \tag{58}$$

We also rigorously prove that $(D_3 + D_1^{\mathrm{T}} D_2^{-1} D_1)^{-1}$ and $(D_2 + D_1 D_3^{-1} D_1^{\mathrm{T}})^{-1}$ are both positive definite, which justifies the appearance of the inverse in eqns (58).

4. IDENTICAL EQUATIONS

Let $\mathbf{n}(\omega)$, $\mathbf{m}(\omega)$ be, respectively, the unit vectors tangent and normal to the interface boundary; we have

$$\mathbf{n}^{\mathrm{T}}(\omega) = (\cos\omega, \sin\omega, 0), \quad \mathbf{m}^{\mathrm{T}}(\omega) = (-\sin\omega, \cos\omega, 0). \tag{59}$$

Therefore, eqns (17) can be generalized by

$$\left.\begin{aligned} R(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e^{\mathrm{T}}_{irs} & -\varepsilon_{is} \end{bmatrix} n_i m_s \\ Q(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e^{\mathrm{T}}_{irs} & -\varepsilon_{is} \end{bmatrix} n_i n_s \\ T(\omega) &= \begin{bmatrix} C_{ijrs} & e_{sji} \\ e^{\mathrm{T}}_{irs} & -\varepsilon_{is} \end{bmatrix} m_i m_s \end{aligned}\right\}. \tag{60}$$

We see that (60) reduces to (17) when $\omega = 0$. Next we consider the generalized eigenrelation

$$N(\omega)\xi = p(\omega)\xi \tag{61}$$

$$N(\omega) = \begin{bmatrix} N_1(\omega) & N_2(\omega) \\ N_3(\omega) & N_1^{\mathrm{T}}(\omega) \end{bmatrix} \quad \xi = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} \tag{62}$$

$$\left.\begin{aligned} N_1(\omega) &= -T^{-1}(\omega)R^{\mathrm{T}}(\omega), N_2(\omega) = T^{-1}(\omega) \\ N_3(\omega) &= R(\omega)T^{-1}(\omega)R^{\mathrm{T}}(\omega) - Q(\omega) \end{aligned}\right\}. \tag{63}$$

It can be proved that the eigenvalues $p(\omega)$ are related to p in eqn (18) by

$$p(\omega) = (p\cos\omega - \sin\omega)/(p\sin\omega + \cos\omega). \tag{64}$$

As before, eqns (61) have eight eigenvalues $p_\alpha(\omega)$, Im $(p_\alpha(\omega)) > 0$, $\alpha = 1, 2, 3$ and 4, which come in four pairs of complex conjugates and can be combined into one compact form as

$$N(\omega)\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{bmatrix} A & P(\omega) \\ B & P(\omega) \end{bmatrix}, \tag{65}$$

where A and B are defined in (21) and

$$P(\omega) = \mathrm{diag}(p_1(\omega), p_2(\omega), p_3(\omega), p_4(\omega)). \tag{66}$$

If we postmultiply both sides of (65) by the matrix $[B^{\mathrm{T}}, A^{\mathrm{T}}]$ and use (22), we have

$$2\begin{bmatrix} AP(\omega)B^{\mathrm{T}} & AP(\omega)A^{\mathrm{T}} \\ BP(\omega)B^{\mathrm{T}} & BP(\omega)A^{\mathrm{T}} \end{bmatrix} = N(\omega)\begin{bmatrix} I - \mathrm{i}S & -\mathrm{i}H \\ \mathrm{i}L & I - \mathrm{i}S^{\mathrm{T}} \end{bmatrix}. \tag{67}$$

Substituting $N(\omega)$ from (62), we obtain the identities

$$\left.\begin{aligned} 2AP(\omega)B^{\mathrm{T}} &= N_1(\omega) - \mathrm{i}[N_1(\omega)S - N_2(\omega)L] \\ &= N_1(\omega) - \mathrm{i}[SN_1(\omega) + HN_3(\omega)L] \\ 2AP(\omega)A^{\mathrm{T}} &= N_2(\omega) - \mathrm{i}[N_1(\omega)H + N_2(\omega)S^{\mathrm{T}}] \\ 2BP(\omega)B^{\mathrm{T}} &= N_3(\omega) - \mathrm{i}[N_3(\omega)S - N_1^{\mathrm{T}}(\omega)L] \end{aligned}\right\}. \tag{68}$$

The proof parallels that of Barnett and Lothe (1973) for the anisotropic elasticity problem, so we have an alternative expression for S, H and L defined in (22)

$$\left.\begin{aligned} S &= \frac{1}{\pi}\int_0^\pi N_1(\omega)\,\mathrm{d}\omega \\ H &= \frac{1}{\pi}\int_0^\pi N_2(\omega)\,\mathrm{d}\omega \\ L &= -\frac{1}{\pi}\int_0^\pi N_3(\omega)\,\mathrm{d}\omega \end{aligned}\right\} \tag{69}$$

for anisotropic piezoelectric materials. The three matrices S, H and L can be used to obtain the real-form solutions of the electroelastic fields in an anisotropic piezoelectric medium, without determining the eigenvectors A and B.

5. FIELDS ALONG THE INTERFACE BOUNDARY

Let $\mathbf{n}(\omega)$, $\mathbf{m}(\omega)$ be, respectively, the unit vectors tangent and normal to the interface boundary, and T_m the stress and electric displacement vectors along the interface boundary. We have

$$T_m = U_{,n} = \cos\omega U_{,1} + \sin\omega U_{,2}. \tag{70}$$

Since $U = U^0$ at the interface boundary, use of (44) leads to

$$T_m = \cos\omega T_2^0 - \sin\omega T_1^0, \tag{71}$$

where T^0 are the stress and the electric displacement vectors in the inclusion. Next consider the stress and the electric displacement vectors normal to the interface boundary, then

$$T_n = -U_{,m} = \sin\omega U_{,1} - \cos\omega U_{,2}. \tag{72}$$

Substituting (39) into this leads to

$$T_n = \sin\omega T_2^\infty + \cos\omega T_1^\infty - 2\mathrm{Re}\{BF_{,m}(z)A^{\mathrm{T}}\}\mathbf{g} - 2\mathrm{Re}\{BF_{,m}(z)B^{\mathrm{T}}\}\mathbf{h}, \tag{73}$$

in which T^∞ are the stress and the electric displacement vectors in the matrix at infinity. The differentiation of (41) and evaluation of the result at the interface boundary (27) yields

$$\begin{aligned} \frac{\partial}{\partial m}\xi_\alpha^{-1} &= (p_\alpha\cos\omega - \sin\omega)\frac{\mathrm{d}}{\mathrm{d}z_\alpha}\xi_\alpha^{-1} \\ &= \left(\frac{1}{a}\cos\omega - \frac{\mathrm{i}}{b}\sin\omega\right)p_\alpha(\omega), \end{aligned} \tag{74}$$

where we have made use of (64). Therefore, using (40) we obtain

$$F_{,m}(z) = \left(\frac{1}{a}\cos\omega - \frac{\mathrm{i}}{b}\sin\omega\right)P(\omega), \tag{75}$$

where $P(\omega)$ is defined in (66). Finally, substituting (75) into (73) and using (68) yields

$$T_n(\omega) = \cos\omega\left\{T_1^\infty - \frac{1}{a}[N_3(\omega)\mathbf{h} + N_1^{\mathrm{T}}(\omega)\mathbf{g}]\right\}$$
$$+\sin\omega\left\{T_2^\infty + \frac{1}{b}[N_3(\omega)(S\mathbf{h}+H\mathbf{g}) + N_1^{\mathrm{T}}(\omega)(S^{\mathrm{T}}\mathbf{g} - L\mathbf{h})]\right\}. \quad (76)$$

If we employ (53) and (54) to eliminate **g** and **h**, an alternative form is given

$$T_n(\omega) = \cos\omega[T_1^\infty + N_3(\omega)(\{\gamma_1^\infty, E_1^\infty\} - \{\gamma_1^0, E_1^0\}) + N_1^{\mathrm{T}}(\omega)(T_2^\infty - T_2^0)]$$
$$+\sin\omega[T_2^\infty + N_3(\omega)(\{\gamma_2^\infty, E_2^\infty\} - \{\gamma_2^0, E_2^0\}) - N_1^{\mathrm{T}}(\omega)(T_1^\infty - T_1^0)], \quad (77)$$

where eqns (76) and (77) are the real-form solution for the elastic and electric fields to the problem of an elliptic inclusion in an infinite piezoelectric medium subject to a uniform stress at infinity. It is clear from (76) and (77) that the coupled fields are only dependent on the identities given by the elastic and electric constants and the boundary conditions. We can also state in the case of a nonpiezoactive medium ($e_{sij} = 0$) there can be no coupled solution; particular formulae of independent elastic and electric fields are derived from the general expressions (76) and (77), which can be exactly reduced to the existing formulae given by Hwu and Ting (1989) using the Stroh method in anisotropic elastic mechanics.

6. CONCLUSION

In this paper, the Stroh method in anisotropic elastic mechanics was used to analyse the coupled elastic and electric fields in an infinite piezoelectric medium containing an elliptical inclusion. The explicit real-form solutions for the electroelastic fields both inside the inclusion and on the boundary of the inclusion and matrix are obtained. The general expression can also be applied in measuring the piezoelectrical constants of piezoelectric composites. It is apparent that understanding of the coupled electrical and mechanical properties of the generalized anisotropic piezoelectric body is essential for the design and manufacture of piezoelectric components.

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APPENDIX

In this Appendix we state the result which is mentioned in Section 2. First, following eqns (18) and (19), we have

$$\begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = p \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}. \tag{A1}$$

If we introduce the matrix $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, which satisfies the identity

$$J = J^{-1} = J^{\mathrm{T}}, \tag{A2}$$

where I is the 4×4 unit matrix, postmultiply both sides of (A1) by the matrix J and use (A2), we obtain

$$N^{\mathrm{T}}(J\xi) = p(J\xi). \tag{A3}$$

Let $\eta = J\xi = \begin{Bmatrix} \mathbf{b} \\ \mathbf{a} \end{Bmatrix}$; eqns (A1) and (A3) can then be rewritten as

$$\left.\begin{aligned} N\xi_\alpha &= p_\alpha \xi_\alpha \\ N^{\mathrm{T}}\eta_\beta &= p_\beta \eta_\beta \end{aligned}\right\} \quad \alpha, \beta = 1, 2, \ldots, 8. \tag{A4}$$

Since the eigenvectors are orthogonal

$$\eta_\beta \xi_\alpha = \xi_\beta^{\mathrm{T}} J \xi_\alpha = \delta_{\beta\alpha}, \tag{A5}$$

use of the 4×4 matrices A and B defined in eqns (21) leads to

$$U = \{\xi_1 \ldots \xi_4, \bar{\xi}_1 \ldots \bar{\xi}_4\} = \begin{bmatrix} A & \bar{A} \\ B & \bar{B} \end{bmatrix}. \tag{A6}$$

It is clear that

$$\left.\begin{aligned} &U^{\mathrm{T}} J U = I \\ &\begin{bmatrix} A & \bar{A} \\ B & \bar{B} \end{bmatrix} \begin{bmatrix} B^{\mathrm{T}} & A^{\mathrm{T}} \\ \bar{B}^{\mathrm{T}} & \bar{A}^{\mathrm{T}} \end{bmatrix} = I \end{aligned}\right\}. \tag{A7}$$

The eqns (A7) can be recast in the expansion form

$$\left.\begin{aligned} AB^{\mathrm{T}} + \overline{AB}^{\mathrm{T}} &= BA^{\mathrm{T}} + \overline{BA}^{\mathrm{T}} = I \\ AA^{\mathrm{T}} + \overline{AA}^{\mathrm{T}} &= BB^{\mathrm{T}} + \overline{BB}^{\mathrm{T}} = 0 \end{aligned}\right\}. \tag{A8}$$

We see that AA^{T} and BB^{T} are pure imaginary matrices. Therefore, the following three matrices introduced by Barnett and Lothe (1973)

$$\left.\begin{aligned} S &= \mathrm{i}(2AB^{\mathrm{T}} - I), \quad H = 2\mathrm{i}AA^{\mathrm{T}} \\ L &= -2\mathrm{i}BB^{\mathrm{T}} \end{aligned}\right\} \tag{A9}$$

can be shown to be real for the electroelastic medium. Now consider the following equation:

$$\begin{bmatrix} S & H \\ -L & S^{\mathrm{T}} \end{bmatrix} = 2\mathrm{i}\begin{Bmatrix} A \\ B \end{Bmatrix}\{B^{\mathrm{T}} \;\; A^{\mathrm{T}}\} - \mathrm{i}I \tag{A10}$$

and rewrite (A8) as

$$\{B^{\mathrm{T}} \;\; A^{\mathrm{T}}\}\begin{Bmatrix} A \\ B \end{Bmatrix} = I. \tag{A11}$$

Then, using (A10) and (A11), we have

$$\begin{bmatrix} S & H \\ -L & S^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} S & H \\ -L & S^{\mathrm{T}} \end{bmatrix} = -I. \tag{A12}$$

This means that

$$\left.\begin{aligned} HL - SS = LH - S^{\mathrm{T}}S^{\mathrm{T}} = I \\ LS + S^{\mathrm{T}}L = SH + HS^{\mathrm{T}} = 0 \end{aligned}\right\} \tag{A13}$$

and the matrices LS and SH are antisymmetric, so we also have the identity

$$HL - SS = I. \tag{A14}$$